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Massive quantum fields in a conical background[1]

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Abstract

Representations of the Klein-Gordon and Dirac propagators are determined in a N dimensional conical background for massive fields twisted by an arbitrary angle $2\pi\sigma$. The Dirac propagator is shown to be obtained from the Klein-Gordon propagator twisted by angles $2\pi\sigma \pm \mathcal{D}/2$ where $\mathcal{D}$ is the cone deficit angle. Vacuum expectation values are determined by a point-splitting method in the proper time representation of the propagators. Analogies with the Aharonov-Bohm effect are pointed out throughout the paper and a conjecture on an extension to fields of arbitrary spin is given.

1 Introduction

Recently interest in quantum fields in a conical spacetime background has been renewed with the possibility of application to the evaluation of geometric entropy [1]. The subject has been studied since the late seventies [2], and investigations became rather intense in the mid-eighties [3-8] due to the importance that cosmic strings may have in cosmology [9]. The literature is mainly concerned with a massless scalar field.

In studying particles and fields on a cone, one meets features which resemble in a great deal those in the Aharonov-Bohm (A-B) set up [10]. The roots of this analogy are in the gauge theory aspects of gravity which are very special in 3 dimensions, as shown in [11] and references therein. Field strengths are concentrated on the symmetry axis, vanishing everywhere else. However, the non vanishing affine connections tell the rest of

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the spacetime that there is a singularity at the symmetry axis. That is the source of the non trivial A-B effect.

By performing a convenient gauge transformation, one can replace the affine connections almost everywhere by an unusual boundary condition [11]. In the A-B situation for example the vector potential can be gauged away everywhere except on some ray. In the new gauge the fields are "twisted" by an angle $2\pi\sigma$,

$$\mathcal{F}(\theta + 2\pi) = \exp\{i2\pi\sigma\}\mathcal{F}(\theta),$$

where θ is the polar angle and σ is the flux parameter [2, 12].

This paper is concerned with massive scalar and spinor fields in a N dimensional conical spacetime. The fields are taken to be "twisted" by an angle $2\pi\sigma$. In section 2 the geometry of a conical spacetime generated by a source carrying spin (spinning cone) is described. The proper time representation of the Klein-Gordon propagator is obtained in section 3. Evaluating the integration over the proper time, another representation is presented. The rest of the paper refers to a spinless cone.

In section 4 the Dirac propagator is expressed in terms of the Klein-Gordon propagator. Its expression is derived using a N-bein for which the spin connection vanishes almost everywhere. This and other resemblances with the A-B effect are also discussed.

Section 5 is concerned with vacuum expectation values. The approach used is the one in [13], which has been used earlier to work out the energy momentum tensor $\langle T^{\mu}{}_{\nu}(x)\rangle$ of a conformal scalar field in a 4 dimensional conical spacetime [3, 14]. All the quantities can be obtained from the twisted Klein-Gordon propagator.

General expressions showing dependence on the radial coordinate ρ are given for the vacuum fluctuations $\langle\phi^2(x)\rangle$ and $\langle T^{\mu}{}_{\nu}(x)\rangle$ of a massive (M) scalar field coupled with the curvature scalar, when $M\rho << 1$. These quantities are evaluated in four dimensions ($N = 4$) showing the dependence on the deficit angle $\mathcal{D}$ and on the twist parameter σ. A mass correction in $\langle\phi^2(x)\rangle$ is determined for a untwisted $\phi(x)$. For spinors, the energy density $\langle T_{00}(x)\rangle$ is given in $N = 4$ when $M\rho << 1$.

The paper ends with some possible extensions of the work. In particular a conjecture on the Feynman propagator of a field of arbitrary spin is given. Throughout the paper $c = \hbar = 1$ and $G = 1/4$.

2 The background

A N dimensional spinning cone [15, 16] is characterized by the Minkowski line element written in cylindrical coordinates

$$ds^2 = d\bar{t}^2 - d\rho^2 - \rho^2 d\varphi^2 - d\mathbf{z}^2, \tag{1}$$

and by the identification

$$(\bar{t}, \rho, \varphi, \mathbf{z}) \sim (\bar{t} + 2\pi S, \rho, \varphi + 2\pi\alpha, \mathbf{z}), \tag{2}$$

where $\mathbf{z} := (z_1, ..., z_{N-3})$, $S \geq 0$ is the spin density of the source on the cone symmetry axis ($\rho = 0$), and $0 < \alpha \leq 1$ is the cone parameter. The latter is related to the deficit angle by $\mathcal{D} = 2\pi(1-\alpha)$ which is proportional to the mass density of the source, $\mu = \mathcal{D}/2\pi$. The unusual identification (2) encapsulates the fact that (1) hides a singularity on the symmetry axis. Clearly when $\alpha = 1$ and $S = 0$ the spacetime becomes the Minkowski one.

3 The Klein-Gordon propagator

The Feynman propagator for a spin 0 field with mass M in the background described above is solution of [17]

$$\left(\Box_x + M^2\right) G_{\mathcal{F}}(x, x') = -\frac{1}{\rho}\delta\left(x - x'\right), \tag{3}$$

where

$$\Box_x = \frac{\partial^2}{\partial \bar{t}^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\mathbf{L}^2}{\rho^2} - \nabla_{\mathbf{z}}$$

with $\mathbf{L} := -i\partial/\partial\varphi$ and $\nabla_{\mathbf{z}} := \sum_{i=1}^{N-3} \partial^2/\partial z_i^2$. Since the spacetime is flat around the cone symmetry axis, the coupling with the curvature scalar does not contribute in (3). If the field is "twisted" by an angle $2\pi\sigma$, it follows from (2) that

$$G_{\mathcal{F}}(\bar{t} + 2\pi S, \varphi + 2\pi\alpha) = \exp\{i2\pi\sigma\} G_{\mathcal{F}}(\bar{t}, \varphi), \tag{4}$$

where the other coordinates have been omitted. The twisted boundary condition (4) implies that the propagator must vanish on the cone symmetry axis for non-integer values of the twist parameter σ, otherwise one gets an inconsistency by shrinking a loop around $\rho = 0$ [18] [3].

If $0 \leq \varphi < 2\pi\alpha$ the δ function in (3) is

$$\delta(x - x') = \delta\left(\bar{t} - \bar{t}' - S(\varphi - \varphi')/\alpha\right)\delta\left(\rho - \rho'\right)\delta(\varphi - \varphi')\delta\left(\mathbf{z} - \mathbf{z}'\right),$$

which is adequate to the boundary condition (4). Since $\Box_x$ is just the D'Alembertian in Minkowski spacetime written in cylindrical coordinates, the non-trivial geometry manifests itself only through (4) [4].

In order to get the proper time representation of $G_{\mathcal{F}}(x, x')$ one needs the eigenfunctions of the operator $\Box_x + M^2$ satisfying (4). They are given by

$$\psi_{\omega,\kappa,m,\mathbf{k}}(x) = \frac{1}{(2\pi)^{(N-1)/2}\alpha^{1/2}} J_{|m+\sigma+\omega S|/\alpha}(\kappa\rho) e^{i[\mathbf{k}\cdot\mathbf{z} - \omega\bar{t} + (m+\sigma+\omega S)\varphi/\alpha]}, \tag{5}$$

where m is a integer, ω and $\mathbf{k}$ are real numbers, κ is a positive real number and J_ν denotes a Bessel function of the first kind. The corresponding eigenvalues are $E_{\omega,\kappa,\mathbf{k}} =$

[3] In fact this is the case only when the propagator is finite at $\rho = 0$. There are other choices of boundary conditions which diverge at $\rho = 0$ [19]. They will not be considered here.

[4] When σ is identified with a flux parameter, the field is taken to be charged and the interaction with the localized magnetic flux is also expressed only through (4).

$\kappa^2 + \mathbf{k}^2 - \omega^2 + M^2$. ωS and σ play the same role in (5), namely to shift the eigenvalues m/α of $\mathbf{L}$. This analogy has also been pointed out in [16, 11] in related contexts.

Using the completeness relation of the Bessel functions

$$\int_0^\infty dk\ kJ_\nu(k\rho)J_\nu(k\rho') = \frac{1}{\rho}\delta(\rho-\rho')$$

and the Fourier expansion (Poisson's formula)

$$\sum_{m=-\infty}^{\infty} \delta(\theta + 2\pi m) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \exp\{im\theta\}\,,$$

one obtains

$$\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\ \int_{-\infty}^{\infty} d\mathbf{k}\int_0^\infty d\kappa\ \kappa\psi_{\omega,\kappa,m,\mathbf{k}}(x)\psi^*_{\omega,\kappa,m,\mathbf{k}}(x') = \frac{1}{\rho}\delta\left(x-x'\right), \tag{6}$$

which is the completeness relation of the eigenfuntions $\psi_{\omega,\kappa,m,\mathbf{k}}(x)$. Considering (6) and by direct application of $\Box_x + M^2$, it can be shown that $G_{\mathcal{F}}(x,x')$ is given by

$$\begin{aligned} G_{\mathcal{F}}^{(N,S,\alpha,\sigma)}(x,x') &= -i\int_0^\infty dT\ \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\ \int_{-\infty}^{\infty} d\mathbf{k} \\ &\quad \times\int_0^\infty d\kappa\ \kappa e^{-iTE_{\omega,\kappa,\mathbf{k}}}\psi_{\omega,\kappa,m,\mathbf{k}}(x)\psi^*_{\omega,\kappa,m,\mathbf{k}}(x'), \end{aligned} \tag{7}$$

where $E_{\omega,\kappa,\mathbf{k}}$ is taken to have an infinitesimal negative imaginary part to make the integration over T converge. $G_{\mathcal{F}}^{(4,0,\alpha,0)}(x,x')$, $G_{\mathcal{F}}^{(4,0,\alpha,1/2)}(x,x')$ and $G_{\mathcal{F}}^{(4,S,\alpha,0)}(x,x')$ reproduce the expressions in [3], [14] and [20] respectively. Evaluating the integrations over $\mathbf{k}$ and κ [21], (7) yields

$$\begin{aligned} G_{\mathcal{F}}^{(N,S,\alpha,\sigma)}(x,x') &= \int_0^\infty dT\ \frac{(T/i\pi)^{1/2}}{\alpha(4\pi iT)^{N/2}}e^{-i\left\{[-(\rho^2+\rho'^2)-(\mathbf{z}-\mathbf{z}')^2]/4T+M^2T\right\}} \\ &\quad \times\int_{-\infty}^{\infty} d\omega\ e^{iT\omega^2-i\omega(t-t')}\sum_{m=-\infty}^{\infty} I_{|m+\sigma+\omega S|/\alpha}\left(\rho\rho'/2iT\right)e^{i(m+\sigma)(\varphi-\varphi')/\alpha}, \end{aligned}$$

where I_ν denotes a modified Bessel function of the first kind, the time coordinate has been redefined [5] as $t := \bar{t} - S\varphi/\alpha$, and the fact that the propagator transforms like a scalar at x and x' (biscalar) has been considered. When the cone is spinless the integration over ω can be performed,

$$\begin{aligned} G_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(x,x') &= \int_0^\infty \frac{dT}{\alpha(4\pi iT)^{N/2}}e^{-i\left\{[(t-t')^2-(\rho^2+\rho'^2)-(\mathbf{z}-\mathbf{z}')^2]/4T+M^2T\right\}} \\ &\quad \times\sum_{m=-\infty}^{\infty} I_{|m+\sigma|/\alpha}\left(\rho\rho'/2iT\right)e^{i(m+\sigma)(\varphi-\varphi')/\alpha}. \end{aligned} \tag{8}$$

[5]In terms of the new time, (1) becomes $ds^2 = (dt + Sd\varphi/\alpha)^2 - d\rho^2 - \rho^2 d\varphi^2 - d\mathbf{z}^2$ and (2) becomes $(t,\rho,\varphi,\mathbf{z}) \sim (t,\rho,\varphi+2\pi\alpha,\mathbf{z})$.

Recalling the Fourier expansion of a plane wave

$$\exp\{z\cos\theta\} = \sum_{m=-\infty}^{\infty} I_{|m|}(z) e^{im\theta},$$

one sees that $G_{\mathcal{F}}^{(N,0,1,0)}(x,x')$ reduces to the proper time expression for the Feynman propagator in the Minkowski spacetime [6],

$$G_{\mathcal{F}}^{(N,0,1,0)}(x,x') = \int_0^\infty \frac{dT}{(4\pi i T)^{N/2}} e^{-i\left[(X-X')^2/4T + M^2 T\right]}, \tag{9}$$

with $X^\mu = (t,\, x = \rho\cos\varphi,\, y = \rho\sin\varphi,\, \mathbf{z})$ being genuine Minkowski coordinates [7]. The usual representation is obtained from (9) noticing that $G_{\mathcal{F}}^{(N,0,1,0)}(x,x') = \int_0^\infty (dT/i(2\pi)^N)$ $\int_{-\infty}^{\infty} d^N K \exp\{-i\left[K(X-X') - (K^2 - M^2)T\right]\}$. Performing the integration over T,

$$G_{\mathcal{F}}^{(N,0,1,0)}(x,x') = \int_{-\infty}^{\infty} \frac{d^N K}{(2\pi)^N} \frac{e^{-iK(X-X')}}{K^2 - M^2 + i\epsilon}. \tag{10}$$

Another representation of $G_{\mathcal{F}}^{(N,S,\alpha,\sigma)}(x,x')$ can be obtained from (7) by performing the integration over the "proper time" T [21] and over $\mathbf{k}$,

$$\begin{aligned} G_{\mathcal{F}}^{(N,S,\alpha,\sigma)}(x,x') &= -\frac{1}{\alpha(2\pi)^{(N+1)/2}} \int_{-\infty}^{\infty} d\omega\ e^{-i\omega(t-t')} \sum_{m=-\infty}^{\infty} e^{i(m+\sigma)(\varphi-\varphi')/\alpha} \\ &\times \int_0^\infty d\kappa\ \kappa J_{|m+\sigma+\omega S|/\alpha}(\kappa\rho) J_{|m+\sigma+\omega S|/\alpha}(\kappa\rho') \frac{\left[\kappa^2 + M^2 - \omega^2\right]^{(N-5)/4}}{\left[(\mathbf{z}-\mathbf{z}')^2\right]^{(N-5)/4}} \\ &\times K_{(N-5)/2}\left(\left\{\left(\kappa^2 + M^2 - \omega^2\right)(\mathbf{z}-\mathbf{z}')^2\right\}^{1/2}\right), \end{aligned}$$

with K_ν denoting a modified Bessel function of the second kind. In the case of a spinless cone,

$$\begin{aligned} G_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(x,x') &= -\frac{i}{\alpha(2\pi)^{N/2}} \sum_{m=-\infty}^{\infty} e^{i(m+\sigma)(\varphi-\varphi')/\alpha} \\ &\times \int_0^\infty d\kappa\ \kappa J_{|m+\sigma|/\alpha}(\kappa\rho) J_{|m+\sigma|/\alpha}(\kappa\rho') \frac{\left[\kappa^2 + M^2\right]^{(N-4)/4}}{\left[(\mathbf{z}-\mathbf{z}')^2 - (t-t')^2\right]^{(N-4)/4}} \\ &\times K_{(N-4)/2}\left(\left\{\left(\kappa^2 + M^2\right)\left[(\mathbf{z}-\mathbf{z}')^2 - (t-t')^2\right]\right\}^{1/2}\right). \end{aligned} \tag{11}$$

$G_{\mathcal{F}}^{(4,0,\alpha,0)}(x,x')$ obtained from (11) agrees with the Wightman function in [8, 22]. So do $G_{\mathcal{F}}^{(3,0,\alpha,0)}(x,x')$ and $G_{\mathcal{F}}^{(3,0,\alpha,1/2)}(x,x')$ agree with the Wightman functions in [23].

[6] This representation resembles a first quantization propagator integrated over the "time" T.

[7] When $\alpha \neq 1$ X^μ are singular "Minkowski" coordinates. In terms of these coordinates the metric tensor is Minkowski everywhere except on $\varphi = 0 \sim 2\pi\alpha$ where X^μ are discontinuous. This is analogous to what happens in the A-B set up.

4 The Dirac propagator

Now a spinless cone will be consider in order to avoid subtleties concerned with unitarity, for the spin 1/2 field. For the same reason σ is not identified with a flux parameter (see, for example [16, 24, 11]).

To study higher spins on the cone one needs to use the N-bein formalism [25, 17]. The metric tensor in (1) can be generated by the N-bein

$$e_a{}^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cos(\varphi/\alpha) & -\rho^{-1}\sin(\varphi/\alpha) & 0 & \cdots & 0 \\ 0 & \sin(\varphi/\alpha) & \rho^{-1}\cos(\varphi/\alpha) & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{12}$$

which is continuous on the ray $\varphi = 0 \sim 2\pi\alpha$. For $N = 3$ $e_a{}^\mu$ is the dreibein in [16] expressed in terms of the coordinate system (t, ρ, φ). For the spin 1/2 field, the spin connection associated with $e_a{}^\mu$ is

$$\begin{aligned} \Gamma_\mu &= \frac{1}{4}\gamma^a\gamma^b e_a{}^\nu e_{b\nu;\mu} \\ &= i\delta_\mu{}^2(\alpha^{-1}-1)\mathcal{J}, \end{aligned} \tag{13}$$

where γ^a are the γ matrices and $\mathcal{J} := i\gamma^1\gamma^2/2$. When the conical singularity is absent, i.e. $\alpha = 1$, Γ_μ vanishes and $e_a{}^\mu$ becomes the usual N-bein associated with the Minkowski coordinate system.

Likewise the vector potential in the A-B set up, the spin connection, can be gauged way by rotating $e_a{}^\mu$ by an angle $[\alpha^{-1} - 1]\varphi$,

$$e_a{}^\mu \to \bar{e}_a{}^\mu = \Lambda_a{}^b([\alpha^{-1}-1]\varphi)e_b{}^\mu,$$

where $\Lambda_a{}^b(\theta)$ is the usual rotation matrix. The new N-bein $\bar{e}_a{}^\mu$ is given by (12) setting $\alpha = 1$ and the new spin connection $\bar{\Gamma}_\mu$ can be obtained from (13). Alternatively one can use the transformation law $\bar{\Gamma}_\mu = U(\theta)\Gamma_\mu U^{-1}(\theta) - U^{-1}(\theta)\partial_\mu U(\theta)$, with [8] $U(\theta) = \exp\{i\theta\mathcal{J}\}$ for $\theta = (\alpha^{-1} - 1)\varphi$, finding that $\bar{\Gamma}_\mu$ vanishes everywhere except on the ray $\varphi = 0 \sim 2\pi\alpha$, where $\bar{e}_a{}^\mu$ is discontinuous (for $\alpha \neq 1$). When $\alpha = 1$, $\bar{e}_a{}^\mu \equiv e_a{}^\mu$, and $\Lambda_a{}^b([\alpha^{-1} - 1]\varphi)$ and $U([\alpha^{-1} - 1]\varphi)$ become the unity transformations.

The spin 1/2 Feynman propagator on the cone satisfies [17]

$$(i\,\not{\nabla}_x - M)\,S_{\mathcal{F}}(x,x') = \frac{1}{\rho}\delta\,(x-x')\,,$$

with $\not{\nabla} := \gamma^a e_a{}^\mu(\partial_\mu + \Gamma_\mu)$, and the boundary condition (4). Thus in terms of $\bar{e}_a{}^\mu$ the new Feynman propagator, $\bar{S}_{\mathcal{F}}(x,x') = U([\alpha^{-1}-1]\varphi)S_{\mathcal{F}}(x,x')U^{-1}([\alpha^{-1}-1]\varphi')$, satisfies the usual equation in Minkowski spacetime with the boundary condition

$$\bar{S}_{\mathcal{F}}(\varphi + 2\pi\alpha) = \exp\{i2\pi\sigma\}U(\mathcal{D})\bar{S}_{\mathcal{F}}(\varphi). \tag{14}$$

[8] $U(\theta)$ is the operator corresponding to a rotation by an angle θ.

It follows that

$$\bar{S}_{\mathcal{F}}(x,x') = (i\ \bar{\partial}\!\!\!/_x + M)\,\bar{G}(x,x') \tag{15}$$

with $\bar{\partial}\!\!\!/ := \gamma^a \bar{e}_a{}^\mu \partial_\mu$ and the bispinor $\bar{G}(x,x')$ satisfying (3) and (14).

To obtain $\bar{G}(x,x')$ one can proceed as in the scalar case. The eigenfunctions of $\Box_x + M^2$ satisfying (14) are given by

$$\begin{aligned}\Psi_{\omega,\kappa,m,\mathbf{k}}(x) \ = \ & \Big\{\frac{1}{2}\left[e^{i(1-\alpha^{-1})\varphi/2}\psi^{+}_{\omega,\kappa,m,\mathbf{k}}(x) + e^{i(\alpha^{-1}-1)\varphi/2}\psi^{-}_{\omega,\kappa,m,\mathbf{k}}(x)\right] \\ & + \left[e^{i(1-\alpha^{-1})\varphi/2}\psi^{+}_{\omega,\kappa,m,\mathbf{k}}(x) - e^{i(\alpha^{-1}-1)\varphi/2}\psi^{-}_{\omega,\kappa,m,\mathbf{k}}(x)\right]\mathcal{J}\Big\}U([\alpha^{-1}-1]\varphi),\end{aligned}$$

where ψ^+ and ψ^- are c-number eigenfunctions of $\Box_x + M^2$ twisted by angles $2\pi\sigma + \mathcal{D}/2$ and $2\pi\sigma - \mathcal{D}/2$ respectively. Then one has that $\bar{G}(x,x')$ is given by (7) replacing ψ and ψ^* by Ψ and $\Psi^\dagger$,

$$\begin{aligned}&\bar{G}^{(N,0,\alpha,\sigma)}(x,x') = U([\alpha^{-1}-1](\varphi-\varphi')) \\ &\times\Big\{\frac{1}{2}\left[e^{i(1-\alpha^{-1})(\varphi-\varphi')/2}G_{\mathcal{F}}^{(N,0,\alpha,\sigma+\mathcal{D}/4\pi)}(x,x') + e^{i(\alpha^{-1}-1)(\varphi-\varphi')/2}G_{\mathcal{F}}^{(N,0,\alpha,\sigma-\mathcal{D}/4\pi)}(x,x')\right] \\ &\quad + \left[e^{i(1-\alpha^{-1})(\varphi-\varphi')/2}G_{\mathcal{F}}^{(N,0,\alpha,\sigma+\mathcal{D}/4\pi)}(x,x') - e^{i(\alpha^{-1}-1)(\varphi-\varphi')/2}G_{\mathcal{F}}^{(N,0,\alpha,\sigma-\mathcal{D}/4\pi)}(x,x')\right]\mathcal{J}\Big\}.\end{aligned} \tag{16}$$

Therefore the Dirac propagator $\bar{S}_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(x,x')$ is determined by the Klein-Gordon propagator $G_{\mathcal{F}}^{(N,0,\alpha,\sigma\pm\mathcal{D}/4\pi)}(x,x')$ through (15). The second term in the curly brackets of (16) is only due to the conical singularity and $\mathcal{D}/4\pi$ plays a role similar to the one played by the flux parameter in the A-B set up.

To get from (16) to the corresponding expressions in terms of $e_a{}^\mu$ and of the polar N-bein $\tilde{e}_a{}^\mu = \Lambda_a{}^b(\varphi/\alpha)e_b{}^\mu$, one drops the factors $U([\alpha^{-1}-1](\varphi-\varphi'))$ and $U(-(\varphi-\varphi'))$ respectively.

Clearly when $\alpha = 1$ ($\mathcal{D} = 0$) and $\sigma = 0$, (16) collapses into (10) and (15) yields the familiar Dirac propagator in Minkowski spacetime.

5 Vacuum expectation values

Propagators can be used to work out vacumm expectation values of physical quantities since the former are defined as the vacuum expectation values of products of operators [17, 26]. In the following some vacuum expectation values are determined. It is assumed that the cone is spinless.

A simple example is the vacuum fluctuation of a scalar field

$$\langle\phi^2(x)\rangle = i\lim_{x'\to x} G_{\mathcal{F}}(x,x'), \tag{17}$$

which obviously diverges. On a conical background however, because the divergences have the same nature as the Minkowski ones, renormalization can be performed simply by subtracting the untwisted contribution in Minkowski spacetime ($\alpha = 1$, $\sigma = 0$) [9].

[9] In [14] renormalization is effected by subtracting the contribution for $\alpha = 1$ and $\sigma = 1/2$. In so doing the twist Casimir effect is missed. It should be recalled that σ can be thought of as a flux parameter.

In order to evaluate (17) one sets in (8) $t = t'$, $\rho = \rho'$, $\mathbf{z} = \mathbf{z}'$ and performs the integration over T [27],

$$\begin{aligned} G_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta) = & \frac{1}{i\alpha(2\pi^{1/2})^N \rho^{N-2}} \times \\ & \sum_{m=-\infty}^{\infty} e^{i(m+\sigma)\Delta/\alpha} \left\{ \frac{\Gamma\left[(N-2)/2 + |m+\sigma|/\alpha\right]\Gamma\left[(3-N)/2\right]}{\pi^{1/2}\Gamma\left[(4-N)/2 + |m+\sigma|/\alpha\right]} \times \right. \\ & {}_1F_2\left[(3-N)/2; (4-N)/2 - |m+\sigma|/\alpha, (4-N)/2 + |m+\sigma|/\alpha; (M\rho)^2\right] \\ & + 2^{-2|m+\sigma|/\alpha}(M\rho)^{N-2(1-|m+\sigma|/\alpha)} \frac{\Gamma\left[(2-N)/2 - |m+\sigma|/\alpha\right]}{\Gamma\left[1+|m+\sigma|/\alpha\right]} \times \\ & \left. {}_1F_2\left[1/2 + |m+\sigma|/\alpha; |m+\sigma|/\alpha + N/2, 1 + 2|m+\sigma|/\alpha; (M\rho)^2\right] \right\}, \end{aligned} \tag{18}$$

where $\Delta := \varphi - \varphi'$ and ${}_1F_2\,[a; b, c; z]$ denotes the generalized hypergeometric function, which converges for all values of z [28].

By taking $M\rho \to 0$ in (18),

$$D_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta) = \frac{\Gamma\left[(3-N)/2\right]}{i\alpha(2\pi^{1/2})^N \pi^{1/2}\rho^{N-2}} \sum_{m=-\infty}^{\infty} \frac{\Gamma\left[(N-2)/2 + |m+\sigma|/\alpha\right]}{\Gamma\left[(4-N)/2 + |m+\sigma|/\alpha\right]} e^{i(m+\sigma)\Delta/\alpha}, \tag{19}$$

which is the massless propagator. Therefore in the regime $M\rho << 1$, the vacuum fluctuations behave approximately as the massless ones. Later it will be seen that this is also the case for the energy momentum tensor. This fact is also mentioned in [22] for $N = 4$.

A superficial inspection of (18) and (19) reveals the usual divergence for $N = 2$ when σ is an integer and another for $N = 3$. In (19) for example, the $N = 2$ divergence is in the term corresponding to $m = -\sigma$ which is an infinite constant. For $N = 3$, $\Gamma(0)\delta(\Delta)$ arises in (19) which can be regularized by considering that $\Gamma[a+z]/\Gamma[b+z] = B[z+a, b-a]/\Gamma[b-a]$. By using the usual integral representation of the beta function $B[z, w]$, the sum in (19) can be evaluated giving a integral representation for $D_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta)$ which is finite for $N = 3$, diverging however for $N = 4$. This representation will not be given here [10].

From (17) and (19), one sees that $\langle\phi^2(x)\rangle^{(N,0,\alpha,\sigma)}$ behave as $1/\rho^{N-2}$. $\langle\phi^2(x)\rangle^{(4,0,\alpha,\sigma)}$ will now be determined, exhibiting the dependence on σ and α. Taking $|\sigma| \leq 1$ is enough to cover all the possibilities in (4). Setting $N = 4$ in (19),

$$\begin{aligned} D_{\mathcal{F}}^{(4,0,\alpha,\sigma)}(\Delta) &= \frac{i}{2(2\pi\alpha\rho)^2} \sum_{m=-\infty}^{\infty} |m+\sigma| e^{i(m+\sigma)\Delta/\alpha} \\ &= \frac{i}{2(2\pi\alpha\rho)^2} e^{i\sigma\Delta/\alpha} \left[|\sigma| + 2\sum_{m=1}^{\infty} m\cos\frac{m\Delta}{\alpha} + 2i\sigma \sum_{m=1}^{\infty} \sin\frac{m\Delta}{\alpha} \right] \\ &= \frac{i}{2(2\pi\alpha\rho)^2} e^{i\sigma\Delta/\alpha} \left[|\sigma| - \frac{1}{2}\csc^2\frac{\Delta}{2\alpha} + i\sigma\cot\frac{\Delta}{2\alpha} \right], \end{aligned} \tag{20}$$

[10] Alternatively another representation for $G_{\mathcal{F}}$ can be used. For example representation (11) for $G_{\mathcal{F}}^{(3,0,\alpha,0)}(x, x')$ and $G_{\mathcal{F}}^{(3,0,\alpha,1/2)}(x, x')$ is used in [23].

which holds for $|\sigma| \leq 1$. $D_{\mathcal{F}}^{(4,0,\alpha,0)}(\Delta)$ and $D_{\mathcal{F}}^{(4,0,\alpha,1/2)}(\Delta)$ reproduce the results in [3] and in [14] respectively. Expanding (20) in powers of Δ, one obtains from (17) after renormalization

$$\langle \phi^2(x) \rangle^{(4,0,\alpha,\sigma)} = -\frac{1}{48\pi^2\rho^2} \left\{ \alpha^{-2} \left[6|\sigma|(1-|\sigma|) - 1 \right] + 1 \right\}. \tag{21}$$

$\langle \phi^2(x) \rangle^{(4,0,\alpha,0)}$ and $\langle \phi^2(x) \rangle^{(4,0,\alpha,1/2)}$ agree with the results in [8] and $\langle \phi^2(x) \rangle^{(4,0,1,1/2)}$ agrees with the result in [29]. The absolute value of σ indicates that the direction of the twist (magnetic flux [11]) is irrelevant. The divergence at $\rho = 0$ is due to the conical singularity and/or the twist [8].

In order to obtain mass corrections to (21) one has to keep terms of order $(M\rho)^2$ in (18). Only the untwisted case will be considered (i.e., $\sigma = 0$). For $N = 4$ divergences arise in the $(M\rho)^2$ term. A convenient procedure to deal with such divergences is dimensional regularization since (18) is expressed in arbitrary dimensions. Setting $N = 4 - \epsilon$, one expands the coefficient of $(M\rho)^2$ in powers of ϵ, isolating the singularities $1/\epsilon$ which cancel out. Then taking $\epsilon \rightarrow 0$,

$$\begin{aligned} G_{\mathcal{F}}^{(4,0,\alpha,0)}(\Delta) &= -\frac{i}{4(2\pi\alpha\rho)^2} \csc^2 \frac{\Delta}{2\alpha} \\ &\quad -\frac{iM^2}{8\alpha\pi^2} \left[\log M\rho + \alpha \log \sin \frac{\Delta}{2\alpha} + \gamma - \frac{1}{2} + (\alpha - 1) \log 2 \right], \end{aligned}$$

where γ is the Euler constant and for which $G_{\mathcal{F}}^{(4,0,1,0)}(\Delta)$ reproduces the corresponding expansion for the massive Klein-Gordon propagator in Minkowski spacetime. Then (17) gives

$$\begin{aligned} \langle \phi^2(x) \rangle^{(4,0,\alpha,0)} &= \frac{1}{48\pi^2\rho^2} \left(\alpha^{-2} - 1 \right) \\ &\quad + \frac{M^2}{8\pi^2} \left[\left(\alpha^{-1} - 1 \right) \left(\log \frac{M\rho}{2} + \gamma - \frac{1}{2} \right) - \log \alpha \right], \end{aligned} \tag{22}$$

reducing to (21) when $M\rho \rightarrow 0$. The massless term in (22) diverges positively at the conical singularity, whereas the massive term diverges negatively.

A similar procedure can be used to obtain the vacuum expectation value of the energy momentum tensor. One can show that in general

$$\langle T^\mu{}_\nu(x) \rangle = i \lim_{x' \rightarrow x} \mathrm{D}^\mu{}_\nu(x, x') G_{\mathcal{F}}(x, x'), \tag{23}$$

where

$$\mathrm{D}^\mu{}_\nu(x, x') := (1 - 2\xi) \nabla^\mu \nabla_{\nu'} + (2\xi - 1/2) \delta^\mu{}_\nu \nabla_\sigma \nabla^{\sigma'} - 2\xi \nabla^\mu \nabla_\nu + 2M^2 \delta^\mu{}_\nu (1/4 - \xi),$$

corresponds to the classical energy momentum tensor of a scalar field coupled to the curvature scalar with coupling parameter ξ [17, 26]. The prime in $\nabla_{\nu'}$ indicates that the

[11] $\phi(x)$ is hermitian. The expectation values of a charged scalar field are twice the ones of $\phi(x)$ [6].

covariant derivative is taken with respect to $x^{\nu'}$. Though the coupling with the curvature scalar does not affect the field equation (3), it does affect $\langle T^\mu{}_\nu(x)\rangle$ through ξ, as is remarked in [8].

One can express (23) for a conical background and with $M\rho << 1$ in terms of $D_{\mathcal{F}}(\Delta)$ and $\partial_\varphi^2 D_{\mathcal{F}}(\Delta)$ only. Using the relations

$$\begin{aligned}
&\lim_{x'\to x}\partial_\varphi\partial_{\varphi'}G_{\mathcal{F}}(x,x') = -\lim_{\Delta\to 0}\partial_\varphi^2 G_{\mathcal{F}}(\Delta) \\
&\lim_{x'\to x}\partial_{z_i}\partial_{z_i'}G_{\mathcal{F}}(x,x') = -\lim_{x'\to x}\partial_{z_i}^2 G_{\mathcal{F}}(x,x') = -\lim_{x'\to x}\partial_t\partial_{t'}G_{\mathcal{F}}(x,x') = \lim_{x'\to x}\partial_t^2 G_{\mathcal{F}}(x,x'),
\end{aligned} \tag{24}$$

which can be derived from (8), and

$$\begin{aligned}
\lim_{x'\to x}\partial_\rho G_{\mathcal{F}}(x,x') &= \frac{2-N}{2\rho}\lim_{\Delta\to 0} D_{\mathcal{F}}(\Delta) \\
\lim_{x'\to x}\partial_t\partial_{t'}G_{\mathcal{F}}(x,x') &= \frac{1}{(1-N)\rho^2}\lim_{\Delta\to 0}\left[\left(\frac{N-2}{2}\right)^2 + \partial_\varphi^2\right] D_{\mathcal{F}}(\Delta) \\
\lim_{x'\to x}\partial_\rho\partial_{\rho'}G_{\mathcal{F}}(x,x') &= \frac{1}{(N-1)\rho^2}\lim_{\Delta\to 0}\left[N\left(\frac{N-2}{2}\right)^2 + \partial_\varphi^2\right] D_{\mathcal{F}}(\Delta) \\
\lim_{x'\to x}\partial_\rho^2 G_{\mathcal{F}}(x,x') &= \frac{1}{(N-1)\rho^2}\lim_{\Delta\to 0}\left\{\frac{(N-2)}{2}\left[\frac{N(N-2)}{2}+1\right] - \partial_\varphi^2\right\} D_{\mathcal{F}}(\Delta)
\end{aligned} \tag{25}$$

which can be obtained by differentiating (8), and by evaluating the integration over T taking $M\rho \to 0$ it follows that

$$\begin{aligned}
\langle T^\mu{}_\nu(x)\rangle^{(N,0,\alpha,\sigma)} = \frac{i}{\rho^2}\Bigg[&\frac{1}{1-N}\lim_{\Delta\to 0}\partial_\varphi^2 D_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta)\,\mathrm{diag}(1,1,1-N,1,\ldots,1) \\
&+(N-2)(\xi-\xi_N)\lim_{\Delta\to 0} D_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta)\,\mathrm{diag}(2-N,1,1-N,2-N,\ldots,2-N)\Bigg],
\end{aligned} \tag{26}$$

with $\xi_N := (N-2)/4(N-1)$. Thus $\langle T^\mu{}_\nu(x)\rangle$ behaves as $1/\rho^N$, is covariantly conserved and traceless when $\xi = \xi_N$. For $N = 4$ and $N = 3$ respectively, the results in [3] and [23] are in agreement with (24), (25) and (26).

To demonstrate the dependence on σ and α, $\langle T^\mu{}_\nu(x)\rangle^{(4,0,\alpha,\sigma)}$ is given by,

$$\begin{aligned}
\langle T^\mu{}_\nu(x)\rangle^{(4,0,\alpha,\sigma)} = \frac{1}{1440\pi^2\rho^4}\Big\{&\Big\{\alpha^{-4}\left[30\sigma^2(1-|\sigma|)^2-1\right]+1\Big\}\,\mathrm{diag}(1,1,-3,1) \\
&+10(6\xi-1)\Big\{\alpha^{-2}\left[6|\sigma|(1-|\sigma|)-1\right]+1\Big\}\,\mathrm{diag}(2,-1,3,2)\Big\},
\end{aligned} \tag{27}$$

where (20) has been used. This result is in agreement with [5] and [7]. $\langle T^\mu{}_\nu(x)\rangle^{(4,0,\alpha,0)}$ has been evaluated in the literature by different methods [3, 4, 8], and $\langle T^\mu{}_\nu(x)\rangle^{(4,0,1,1/2)}$ and $\langle T^\mu{}_\nu(x)\rangle^{(4,0,\alpha,1/2)}$ have been determined in [29] and [8] respectively. Clearly mass corrections to (27) can be obtained in the same way as in (22). They will not be given here.

For the spin 1/2 field where the propagators are defined in terms of anticommuting fields, it can be shown that the vacuum expectation value corresponding to the classical

energy momentum tensor [17] is

$$\langle T_{\mu\nu}(x)\rangle = \frac{1}{4}\lim_{x'\to x} \mathrm{Tr}\left[\bar{\gamma}_\mu(\partial_\nu - \partial_{\nu'}) + \bar{\gamma}_\nu(\partial_\mu - \partial_{\mu'})\right] \bar{S}_{\mathcal{F}}(x,x'),$$

where $\bar{\gamma}_\mu = \bar{e}_{a\mu}\gamma^a$. For $M\rho << 1$ a general expression such as (26) can also be obtained from (24) and (25). It will not be given here. Only the energy density $\langle T_{00}(x)\rangle^{(4,0,\alpha,\sigma)}$ will be given. Using (15), (16) and (8),

$$\langle T_{00}(x)\rangle^{(N,0,\alpha,\sigma)} = i\lim_{x'\to x} \mathrm{Tr}\left\{\partial_t^2 \bar{G}^{(N,0,\alpha,\sigma)}(x,x')\right\}.$$

Observing the properties of the trace of the γ matrices, and also (24), (25) and (20),

$$\langle T_{00}(x)\rangle^{(4,0,\alpha,\sigma)} = -\frac{1}{12\pi^2\rho^4}\left\{\frac{\alpha^{-4}}{2}\left[\sigma_+^2(1-|\sigma_+|)^2 + \sigma_-^2(1-|\sigma_-|)^2 - \frac{1}{15}\right]\right.$$
$$\left. +\alpha^{-2}\left[|\sigma_+|(1-|\sigma_+|) + |\sigma_-|(1-|\sigma_-|) - \frac{1}{3}\right] + \frac{11}{30}\right\}$$

where $\sigma_\pm := \sigma \pm \mathcal{D}/4\pi$ and which holds for $|\sigma_\pm| \le 1$. Two cases can be of particular interest, the Casimir effect due to the conical singularity only

$$\langle T_{00}(x)\rangle^{(4,0,\alpha,0)} = -\frac{1}{2880\pi^2\rho^4}(\alpha^{-2}-1)(7\alpha^{-2}+17)$$

in agreement with [5], and

$$\langle T_{00}(x)\rangle^{(4,0,1,\sigma)} = \frac{1}{12\pi^2\rho^4}|\sigma|(\sigma^2-1)(2-|\sigma|),$$

which is the Casimir effect caused by the twist only.

6 Conclusion and remarks

Summarizing, representations of the Feynman propagator $G_{\mathcal{F}}^{(N,S,\alpha,\sigma)}(x,x')$ of a massive scalar field (M) twisted by an angle $2\pi\sigma$ have been determined in a background of a N dimensional spinning cone (S) of deficit angle $\mathcal{D} = 2\pi(1-\alpha)$. These representations are a generalization of those in the literature. It has been shown that the Dirac propagator $S_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(x,x')$ can be obtained from $G_{\mathcal{F}}^{(N,0,\alpha,\sigma\pm\mathcal{D}/4\pi)}(x,x')$. The calculation of $S_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(x,x')$ has been performed in a gauge in which the spin connection vanishes everywhere except on a ray. Various Aharonov-Bohm like features as this one have been discussed.

In the coincidence limit $t=t'$, $\rho=\rho'$, $\mathbf{z}=\mathbf{z}'$, $G_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta)$ has been expressed in terms of series in powers of $M\rho$ reducing, when $M\rho << 1$, to the massless expression $D_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta)$, which behaves as $1/\rho^{N-2}$. Using this series representation, vacuum expectation values have been evaluated. $\langle\phi^2(x)\rangle^{(4,0,\alpha,\sigma)}$ has been obtained and the mass correction in $\langle\phi^2(x)\rangle^{(4,0,\alpha,0)}$ has shown to have a logarithmic divergence at the conical singularity.

The coincidence limit of the derivatives of $G_{\mathcal{F}}^{(N,0,\alpha,\sigma)}(\Delta)$ in the formulas for the energy momentum tensors $\langle T^{\mu}{}_{\nu}(x)\rangle^{(N,0,\alpha,\sigma)}$ have been written in terms of $D_{\mathcal{F}}(\Delta)$ and $\partial_{\varphi}^2 D_{\mathcal{F}}(\Delta)$ when $M\rho << 1$. As a consequence, a general expression for $\langle T^{\mu}{}_{\nu}(x)\rangle^{(N,0,\alpha,\sigma)}$ of a scalar field has been given. It behaves as $1/\rho^N$, satisfies the covariant conservation law and is traceless for the conformal coupling with the curvature scalar. It has been remarked that a similar general expression can also be obtained for spinors. $\langle T^{\mu}{}_{\nu}(x)\rangle^{(4,0,\alpha,\sigma)}$ of a scalar field and $\langle T_{00}(x)\rangle^{(4,0,\alpha,\sigma)}$ of a spinor field have been evaluated, showing the dependence on the parameters σ and α.

Comparing the expressions in section 3 with those in section 4, one sees a loose correspondence, $\omega \times S \leftrightarrow \mathcal{D}/4\pi = \mu \times 1/2$. In the context of first quantization in three dimensions ($N = 3$), it has been found [16] that $\omega \leftrightarrow \mu$ and $S \leftrightarrow s$, where s is the spin of the particle. If these correspondences are carried over to the context of second quantization, one concludes that the factor $1/2$ above refers to the spin of the field. In the light of this remark and that of [7] one is led to conjecture that formula (16) also holds for arbitrary spins, replacing $G_{\mathcal{F}}^{(N,0,\alpha,\sigma\pm\mathcal{D}/4\pi)}(x,x')$ by $G_{\mathcal{F}}^{(N,0,\alpha,\sigma\pm\mathcal{D}s/2\pi)}(x,x')$ and with $\mathcal{J}$ being the spin s generator of a rotation. The γ matrices would be replaced by the ones mentioned in [7]. Thus the Feynman propagator of a spin s field twisted by an angle $2\pi\sigma$ would be obtained from the scalar propagator twisted by angles $2\pi\sigma \pm \mathcal{D}s$. But this deserves further investigation.

Another extension of this work is to consider $S_{\mathcal{F}}^{(N,S,\alpha,\sigma)}(x,x')$ when $S \neq 0$ and with flux parameter σ. In doing so one must face the problems mentioned earlier concerning unitarity. Also worth investigating is the behaviour of the vacuum expectation values when $S \neq 0$. Using the asymtotic limit of ${}_1F_2\,[a;b,c;z]$ for large z, they can in principle be evaluated also when $M\rho >> 1$. The numerical analysis of the vacuum expectation values of a scalar field in $N = 3$ [23, 30] can be extended to spinors through (16).

Note. After this work was completed references [31, 32] appeared that also consider massive twisted fields in a N dimensional conical background using different methods.

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